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COMMENT

On the Schrödinger equation for the interaction $x^2+\lambda x^2/(1+gx^2)$

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Abstract. The lowest even- and odd-parity energy levels of Schrödinger's equation for the interaction $x^2+\lambda x^2/(1+gx^2)$ for arbitrary positive values of λ and g are obtained accurately from a first-order variational perturbation treatment.

Schrödinger's equation for the potential

$$V(x)=x^2+\lambda x^2/(1+gx^2), \qquad \lambda, g \geqslant 0, \tag{1}$$

has been the subject of many recent calculations. *Exact* solutions of the form

$$\psi(x)=\exp(-x^2/2)\phi(x), \tag{2}$$

where $\phi(x)$ is a polynomial containing either even or odd powers of x, are known to exist when the parameters λ and g are suitably related (Flessas 1981, Varma 1981). For other values of the potential parameters, *approximate* eigenvalues have been calculated by a large number of different methods. (The recent paper of Chaudhuri and Mukherjee (1983) in this journal contains a comprehensive list of references.)

In spite of appearances the approximation method of Chaudhuri and Mukherjee (1983) is actually equivalent to first-order perturbation theory. Moreover, their results may be improved without calculating higher-order corrections, by simply including a variational scale parameter. Several of the earlier treatments also exploited scaling transformations suitable for limited ranges of λ and g (Mitra 1978, Kaushal 1979), or appropriate (approximately) to a secular equation treatment of a *set* of levels (Bessis and Bessis 1980). Here, we treat the lowest even- and odd-parity levels only, so that the scale parameters may be chosen so as to *optimise* the energy through first order.

Basically, we follow the approach of Bessis and Bessis (1980), and choose

$$H_0=-\mathrm{d}^2/\mathrm{d}x^2+\alpha^2x^2 \tag{3}$$

but leave α to be determined. The exact solutions of this model H_0 are well known, and the energy of the nth state may be calculated correct to first order in the perturbation

$$H_1=(1-\alpha^2)x^2+\mu gx^2/(1+gx^2), \qquad \mu=\lambda/g, \tag{4}$$

from the formula

$$E_0^{(n)}(\alpha)+E_1^{(n)}(\alpha)=(n+\tfrac{1}{2})(\alpha+1/\alpha)+\langle \mu gx^2/(1+gx^2)\rangle_n. \tag{5}$$

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The expectation value $\langle \mu g x^2/(1+gx^2)\rangle_n$ may be written in various ways, in particular

$$\langle \mu g x^2/(1+gx^2)\rangle_n = \mu - \mu\langle 1/(1+gx^2)\rangle_n = \tfrac{1}{2}\mu - \tfrac{1}{2}\mu\langle (1-gx^2)/(1+gx^2)\rangle_n. \qquad (5a, b)$$

Bessis and Bessis (1980) based their calculations on (5*a*), while Chaudhuri and Mukherjee (1983) used (5*b*); both integrals may be evalued analytically.

For the lowest even and odd levels, (5) constitutes an *upper bound* to the exact energy, and may be minimised with respect to α. Explicitly, we have

$$E(n=0) \leqslant \tfrac{1}{2}(\alpha + 1/\alpha) + \mu[1 - I_0(t)], \qquad (6a)$$

$$E(n=1) \leqslant \tfrac{3}{2}(\alpha + 1/\alpha) + \mu - 2\mu t^2[1 - I_0(t)], \qquad (6b)$$

where we write (cf Bessis and Bessis 1980)

$$t^2 = \alpha/g, \qquad I_0(t) = \sqrt{\pi}\, t \exp(t^2) \operatorname{erfc} t,$$

and erfc t denotes the *complementary error function*

$$\operatorname{erfc} t = (2/\sqrt{\pi}) \int_t^\infty \exp(-u^2)\, du \qquad (7b)$$

(see e.g., Abramowitz and Stegun 1965).

Introducing a power series representation of the error function yields the following approximations, rapidly convergent for small t (i.e. for large g):

$$E(n=0) = \tfrac{1}{2}(\alpha + 1/\alpha) + \mu(1 - \sqrt{\pi}\, t + 2t^2 - \sqrt{\pi}\, t^3 + \tfrac{4}{3}t^4 + \ldots), \qquad (8a)$$

$$E(n=1) = \tfrac{3}{2}(\alpha + 1/\alpha) + \mu(1 - 2t^2 + 2\sqrt{\pi}\, t^3 - 4t^4 + 2\sqrt{\pi}\, t^5 - \ldots). \qquad (8b)$$

Note that both these approximations differ (in their coefficients of t^2) from the expansions of Chaudhuri and Mukherjee (1983).

For sufficiently large g, the series in t may be neglected and the optimal values of α are clearly close to unity for both levels; more precise optimal values of α may be obtained by retaining the lowest-order terms of (8*a*, *b*), and it is at once apparent that α_{opt} is *not* a simple bilinear function of the parameters λ and g. In particular, for the $n = 1$ level,

$$\alpha_{\text{opt}} = (1 - 4\lambda/3g^2)^{-1/2}. \qquad (9)$$

The series (8*a*, *b*) converge too slowly to be useful for large t (small g), but we may then use an asymptotic expansion for $I_0(t)$ to obtain

$$E(n=0) = \tfrac{1}{2}(\alpha + 1/\alpha) + \mu(1/2t^2 - 3/4t^2 + 15/8t^6 - \ldots), \qquad (10a)$$

$$E(n=1) = \tfrac{3}{2}(\alpha + 1/\alpha) + 3\mu(1/2t^2 - 5/4t^4 + 35/8t^6 - \ldots). \qquad (10b)$$

Retaining only the leading term of the asymptotic expansion yields the optimal scale factor (for both levels)

$$\alpha_{\text{opt}} = (1+\lambda)^{1/2}, \qquad (11)$$

a result which merely confirms that $V(x)$ reduces effectively to a harmonic potential with $\alpha^2 = 1 + \lambda$ when g is sufficiently small.

For g neither large nor small, the first-order upper bounds (6) must be optimised numerically. Table 1 contains some results for a representative set of values of the potential parameters, λ and g. It will be apparent that, with α chosen optimally, a

Table 1. Energies of the lowest even and odd levels.

	λ	0.1		1		10		100	
g		(1)	(2)	(1)	(2)	(1)	(2)	(1)	(2)
0.1	$E(0)$	1.04318 (1.0384†)	1.04317	1.38069 (1.3525)	1.38053	3.25034 (3.1825)	3.25026	9.97650 (9.9765)	9.97618
	$E(1)$	3.12015 (1.0341)	3.12008	4.08100 (1.3333)	4.07988	9.62116 (3.0937)	9.61907	29.77784 (10.0)	29.78119
1	$E(0)$	1.02412 (1.0100)	1.02410	1.23313 (1.1449)	1.23237	2.79285 (2.3870)	2.78233	9.37089 (8.7719)	9.35942
	$E(1)$	3.05152 (1.0100)	3.05149	3.50843 (1.0816)	3.50742	7.45289 (1.8769)	7.41751	26.76264 (8.0)	26.70597
10	$E(0)$	1.00595 (1.0049)	1.00594	1.05938 (1.0240)	1.05930	1.58391 (1.1560)	1.58003	5.89927 (3.4810)	5.79395
	$E(1)$	3.00884 (1.0049)	3.00881	3.08810 (1.0049)	3.08809	3.87980 (1.0240)	3.87904	11.64825 (1.3690)	11.57220
100	$E(0)$	1.00084 (1.0)	1.00084	1.00841 (1.0)	1.00841	1.08409 (1.0100)	1.08406	1.83851 (1.0754)	1.83639
	$E(1)$	3.00098 (1.0)	3.00098	3.00983 (1.0)	3.00983	3.09832 (1.0)	3.09832	3.98317 (1.0100)	3.98310

(1) Present calculation; † denotes optimal α.
(2) From Mitra (1978) or Bessis and Bessis (1980).

single function usually provides high accuracy†. This should be contrasted with the observation of Bessis and Bessis (1980) that overall accuracy of three significant figures requires combinations of four (non-optimally scaled) functions.

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† Our result for $\lambda = 100$, $g = 0.1$ is actually *lower* (and therefore more accurate) than those of Mitra (1978) and Bessis and Bessis (1980).